Shorter Die Battles (and Order Statistics)

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We rederive the main result in [1] in a shorter way, without combinatorics. If we have m independent and identically distributed (i.i.d.) random variables (r.v.'s) $X_1, \ldots, X_m$, with $X_i = X$ having cumulative distribution function (cdf) $F_X(x) = \Pr(X \le x)$, then the r.v. $X_{(m)} = \max_i X_i$ has cdf

$$F_{X_{(m)}}(x) = \Pr(X_{(m)} \le x) = \Pi_{i=1}^{m} \Pr(X_i \le x) = \left(\Pr(X \le x)\right)^m = F_X^m(x).$$

Let X be a uniform r.v. with support $\mathcal{X} = \{1, 2, \ldots, k\}$ modelling a fair die with k faces. For $x \in \mathcal{X}$ the cdf of X is

$$F_X(x) = \frac{x}{k}.$$

If $Y_{(n)}$ is the maximum of the i.i.d. r.v.'s $Y_1, \ldots, Y_n$ with $Y_i = X$ and independent of $X_1, \ldots, X_m$, then

$$\begin{aligned}
\Pr(X_{(m)} > Y_{(n)}) &= \sum_{a>b} \Pr(X_{(m)} = a, Y_{(n)} = b) \\
&= \sum_{a=2}^{k} \Pr(X_{(m)} = a) \Pr(Y_{(n)} < a) \\
&= \sum_{a=2}^{k} \big(F_{X_{(m)}}(a) - F_{X_{(m)}}(a-1)\big) F_{Y_{(n)}}(a-1) \\
&= \sum_{a=2}^{k} \left(F_X^m(a) - F_X^m(a-1)\right) F_X^n(a-1) \\
&= \frac{1}{k^{m+n}} \sum_{a=2}^{k} \left(a^m - (a-1)^m\right) (a-1)^n.
\end{aligned}$$

References

[1] S.J. Miller. Die battles and order statistics. Class Notes from Math 162: Statistics, Brown University, Spring 2006

https://web.williams.edu/Mathematics/sjmiller/public_html/BrownClasses/162/DieBattle.pdf.