Benford's Law: Hammering a Square Peg Into a Round Hole?

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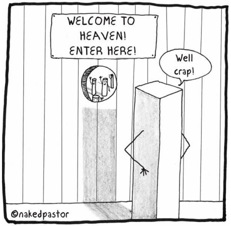
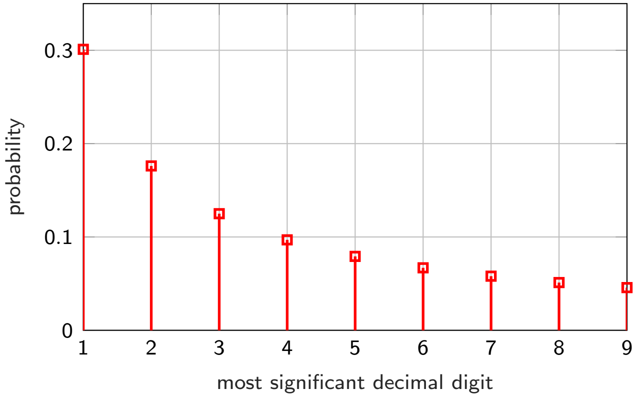


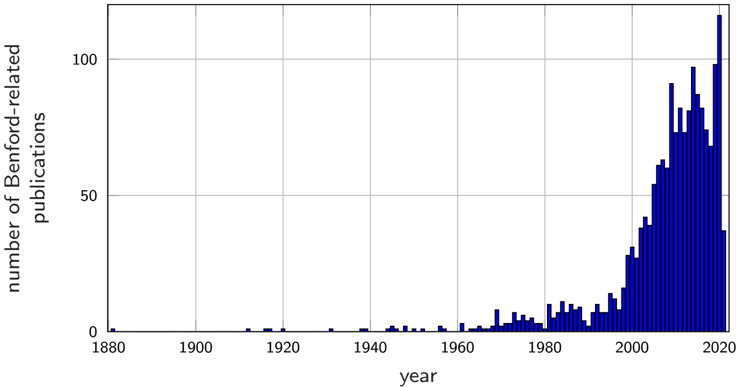
Benford's Law

- Newcomb (1881) and, independently, Benford (1938) noticed the following pattern in certain datasets:



Research on Benford's Law

- The appearance of Benford's distribution in many different scenarios has been extensively studied



total: 1,735 publications

[source: `benfordonline.net`]

Legal Disclaimer

- Many recurrence relations comply exactly with Benford's law
  - Pochhammer numbers, Bell numbers, Fibonacci numbers...
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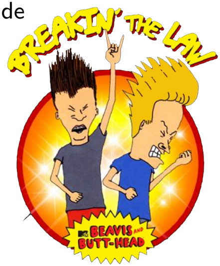

- Should we stop calling Benford's law a "law"?

General Distribution of the k Most Significant b-ary Digits

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- Letting $Y = \log_b X$, the pmf of $A_{(k)}$ can be obtained from the cdf of $\{Y\}$, $F_{\{Y\}}(y) = \Pr(\{Y\} \leq y)$, as follows:

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  - the j-th MSD can also be modelled using $A_{[j]} = A_{(j)} \pmod{b}$

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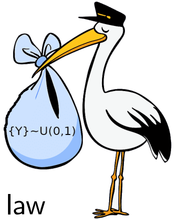
  - the j-th MSD (for $j \geq 2$) is distributed as

$$\Pr(A_{[j]} = a) = \log_b\left(\frac{\Gamma\left((a+1)b^{-1} + b^{j-1}\right)\Gamma\left(ab^{-1} + b^{j-2}\right)}{\Gamma\left((a+1)b^{-1} + b^{j-2}\right)\Gamma\left(ab^{-1} + b^{j-1}\right)}\right)$$

  where $a \in \{0, 1, \ldots, b-1\}$ and $\Gamma(\cdot)$ is the Gamma function
    - this closed-form expression was never previously given

But. . . Where Do Benford r.v.'s Come From?



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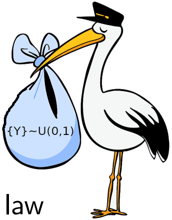

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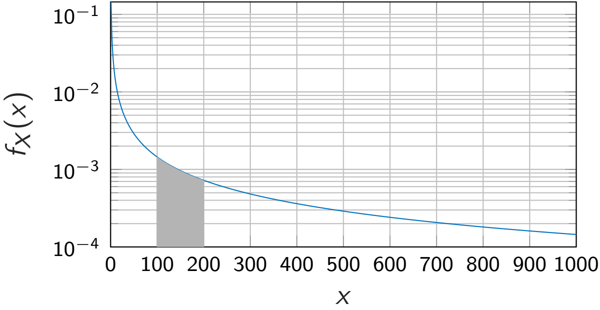

$\Pr\left(X \in (100, 200)\right)$

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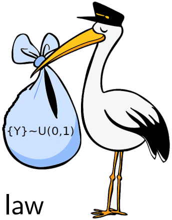

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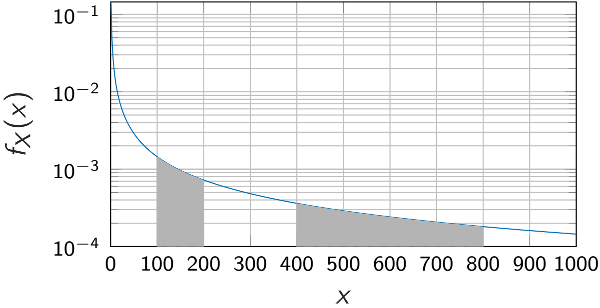

$$\Pr\left(X \in (100, 200)\right) = \Pr\left(X \in 4 \times (100, 200)\right)$$

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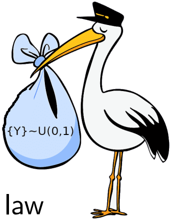

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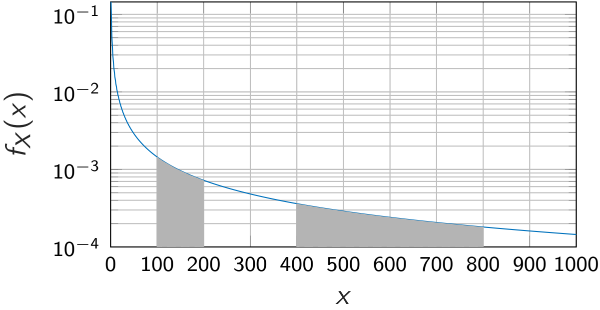

$$\Pr\left(X \in (x', x)\right) = \Pr\left(X \in \alpha\,(x', x)\right) \Rightarrow X \text{ is strictly scale invariant}$$

Strict Scale Invariance and Base Invariance

- Property of the pdf of strictly scale-invariant X

$$\boxed{f_X(x) = \alpha\, f_X(\alpha\, x)} \quad \alpha > 0$$

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- Consequences: Y is uniform, and so X must have finite support which must also depend on b to ensure $\{Y\} \sim U(0,1)$

→ The common notion "*scale-invariant data that follows Benford's law is base invariant*" can only be an approximation

The One and Only, But Often a Misfit

- The pdf of a strictly scale invariant r.v. X must be $\propto x^{-1}$
  $\rightarrow$ the prize-competition distribution is the only choice

$$f_X(x) = \frac{1}{x \ln(x_\mathrm{M}/x_\mathrm{m})}, \quad 0 < x_\mathrm{m} \leq x \leq x_\mathrm{M}$$

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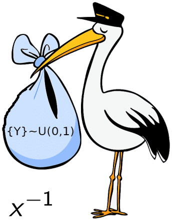

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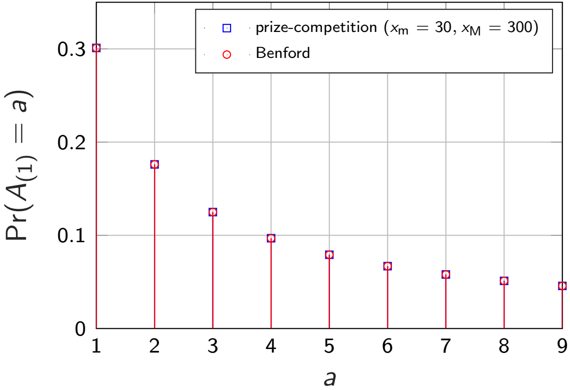
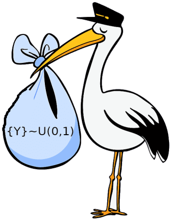
  - plus, for X to be Benford it must hold that $\log_b(x_\mathrm{M}/x_\mathrm{m}) \in \mathbb{Z}$

First Significant Digit in Prize-Competition Distribution





- If $\log_b(x_M/x_m) \in \mathbb{Z}$ we get Benford's distribution

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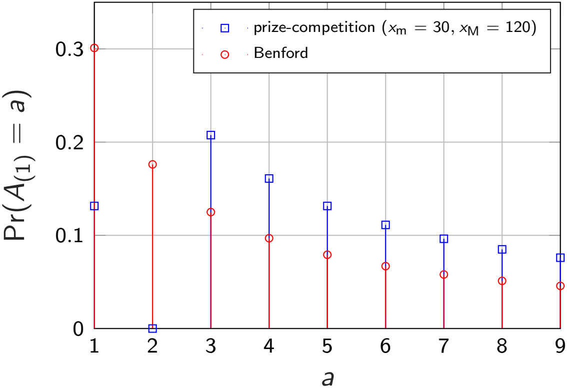

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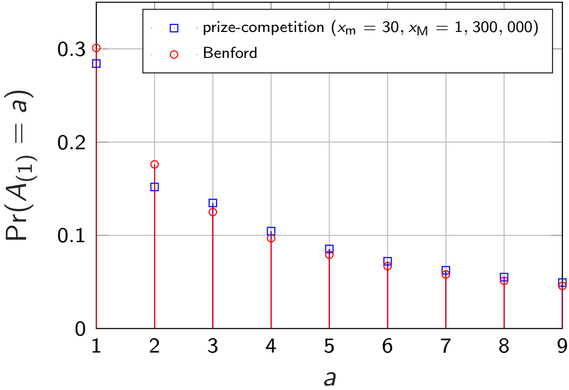

- If $\log_b(x_M/x_m) \notin \mathbb{Z}$ a mismatch is inevitable. . . but it decreases if the pdf spreads over many orders of magnitude
- Still, the prize-competition distribution is relatively uncommon

More Plausible Scale Invariance

- Consider a more relaxed definition of scale invariance:

$$\boxed{f_X(x) = \alpha^{\nu} f_X(\alpha\, x)} \quad \nu > 1$$

$\rightarrow$ The Pareto pdf is the only one to conform to this criterion

$$f_X(x) = \frac{s\, x_{\mathrm{m}}^s}{x^{s+1}}, \quad 0 < x_{\mathrm{m}} \le x,\ s > 0$$

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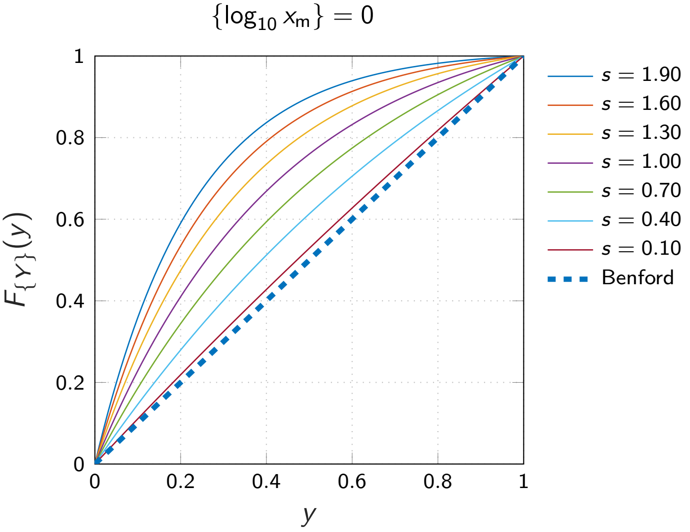
- Relevance: the Central Limit Theorem has a hidden side...
  - "heavy-tailed distributions, such as Pareto, are as prominent as the Gaussian distribution —if not more" (Nair et al., 2021)

cdf of $\{Y\} = \{\log_b X\}$ for Pareto X

s: shape parameter
x_m: minimum value



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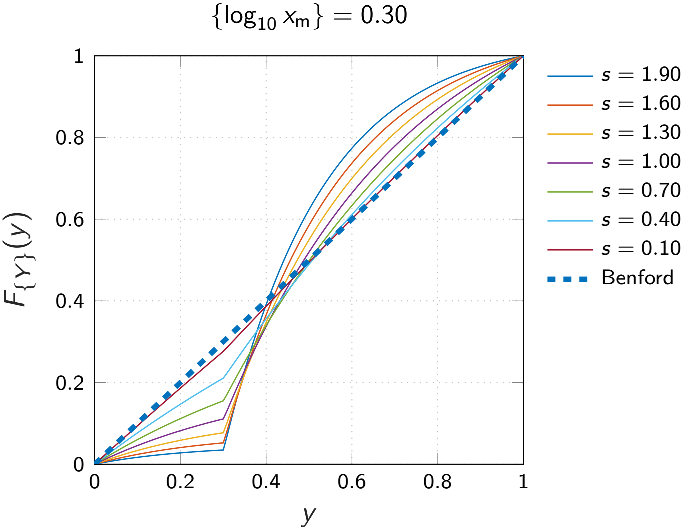

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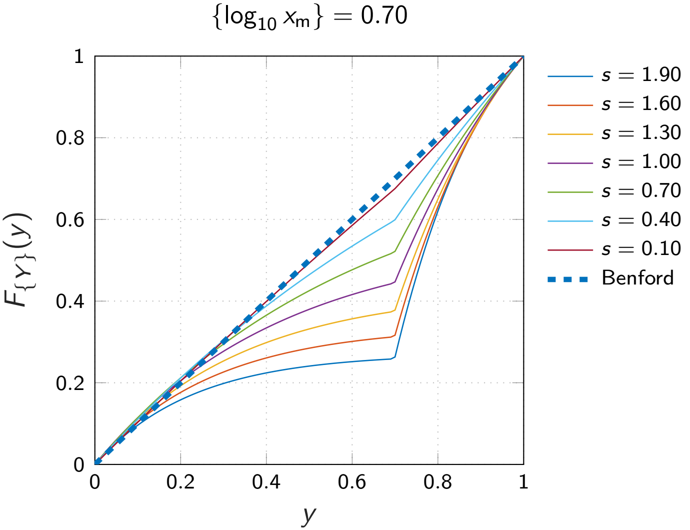

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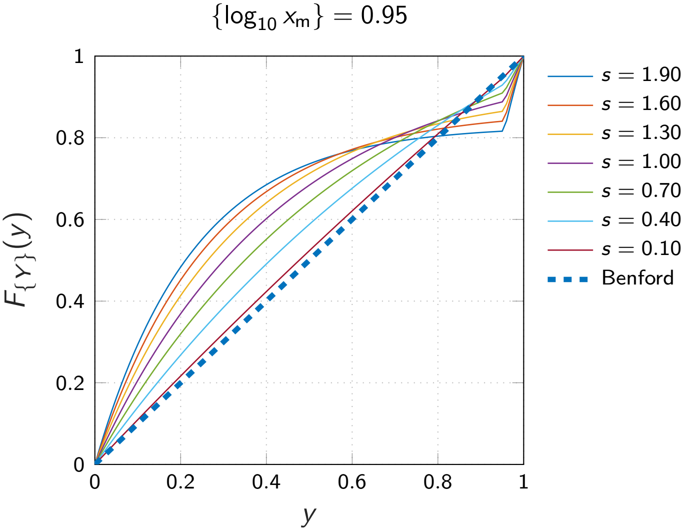

Wrapping it Up

- With the cdf of $\{Y\}$ and the general expression, we get the pmf of the k most significant b-ary digits for a Pareto r.v. X

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where $a \in \mathcal{A}_{(k)}$, $\xi = \{\log_b x_{\mathrm{m}}\} + k - 1$ and $u(\cdot)$ is unit-step function

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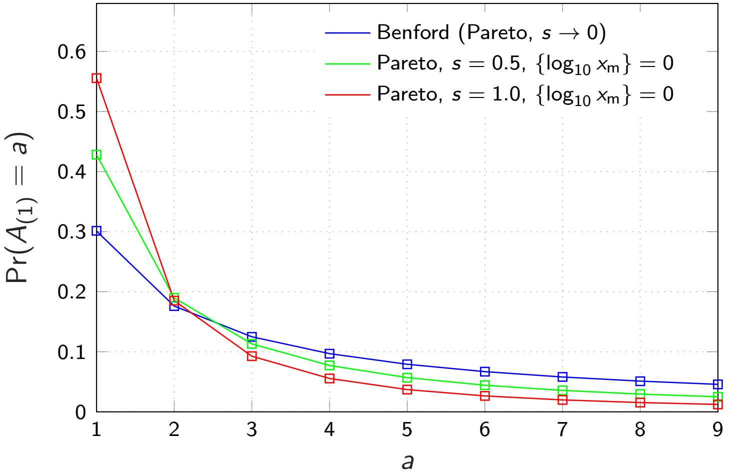
  - as $s \to 0$ the distribution above tends to Benford's
  - **but**: the significant digits of scale-invariant datasets are far more likely to follow this distribution rather than Benford's

Distribution of the k MSDs of a Pareto Variable

- Pseudorandom empiricals vs theoreticals, $k = 1$



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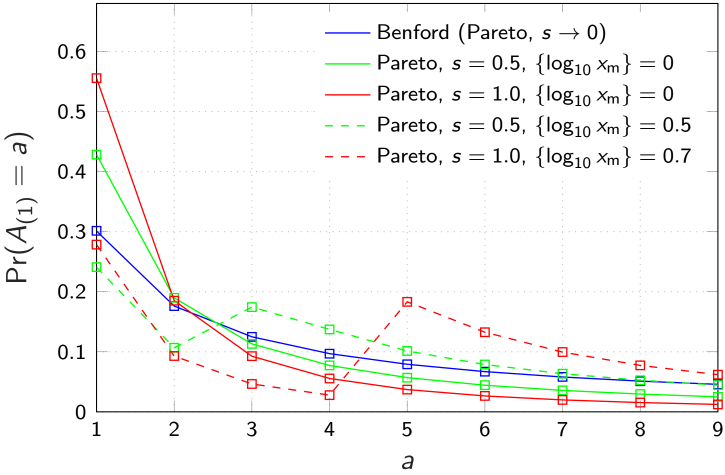

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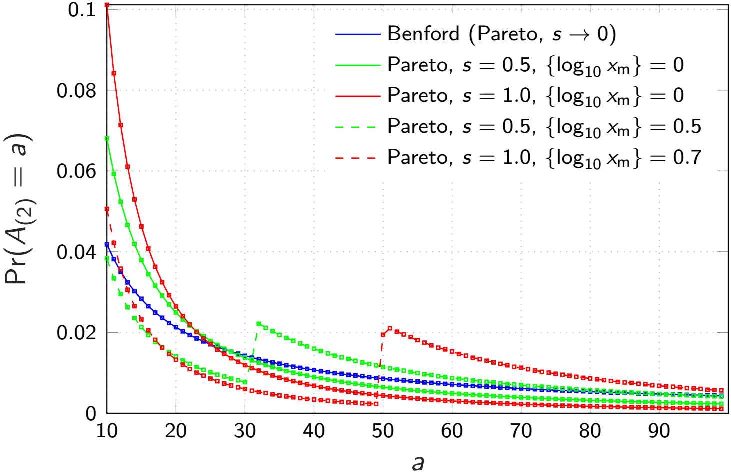

The Butterfly Effect

- Special case $\{\log_b x_\mathrm{m}\} = 0$ (i.e. no kink in the pmf)

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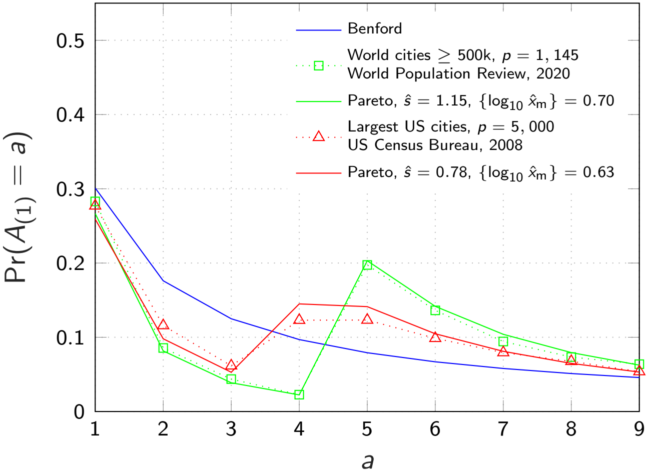
- Identified and named only in 2015, in a Lepidoptera study by Kozubowski et al.: discrete truncated Pareto (DTP) pmf
  - jaw-dropping fact: DTP can be obtained by quantising either
    1. a truncated Pareto r.v.
    2. the fractional part of the logarithm of a standard Pareto r.v.

First Significant Digit in Real Scale-Invariant Datasets

- Scale-invariant datasets are typically assumed to follow Benford's distribution...



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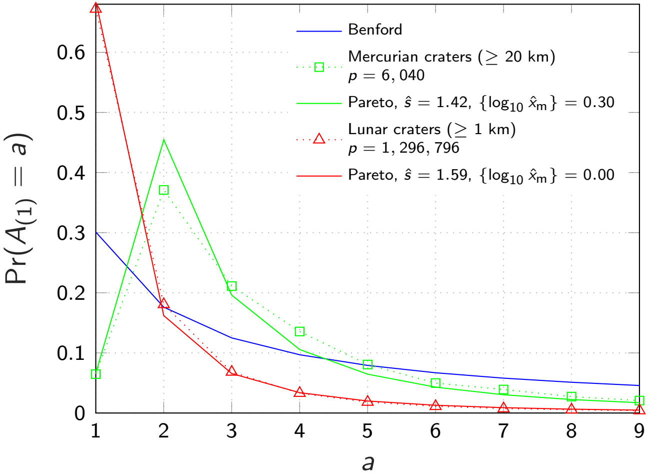

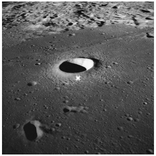

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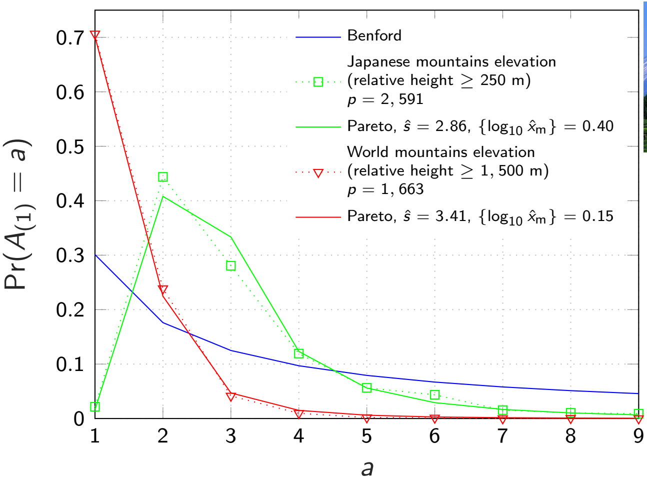

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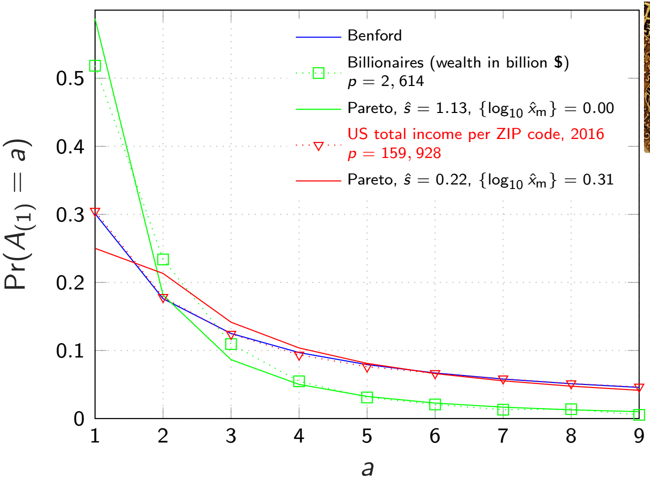

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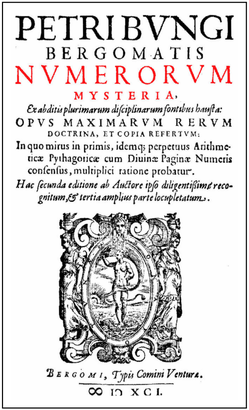
- But: why look at the most significant digits of a set of numbers instead of looking at those numbers themselves?


PETRI BVNGI
BERGOMATIS
NVMERORVM
MYSTERIA,
Ex abditis plurimarum disciplinarum fontibus hausta:
OPVS MAXIMARVM RERVM
DOCTRINA, ET COPIA REFERTVM:
In quo mirus in primis, idemq; perpetuus Arithmeticæ Pythagoricæ cum Diuinæ Paginæ Numeris consensus, multiplici ratione probatur.
Hac secunda editione ab Auctore ipso diligentissime recognitum, & tertia amplius parte locupletatum.
BERGOMI, *Typis Comini Venturæ.*
∞ IↃ XCI.

Chasing Shadows in Forensic Analysis...

- Discrete projection of continuous data → <u>information loss</u>

Plato's allegory of the cave

"light came into the world, and men loved darkness rather than light"

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Your Significant Others: Continued Fraction Coefficients

- Continued fractions (CF): a way of representing numbers alternative to positional base b number systems

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$$Y_0 = [A_0; A_1, A_2, A_3, \ldots]$$

- If $Y_0 = \log_b X$ and X is Benford, then

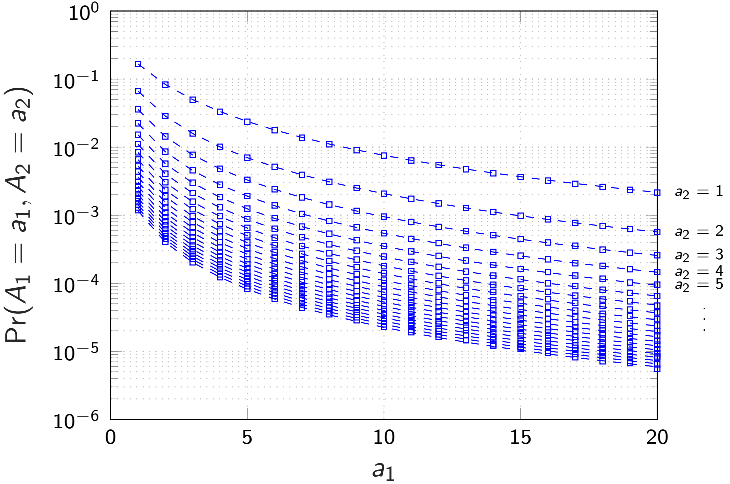
$$\boxed{\begin{aligned}\Pr(A_1 = a_1, \ldots, A_k = a_k) =& (-1)^k \big([0; a_1, \ldots, a_{k-1}, a_k + 1] \\ & - [0; a_1, \ldots, a_{k-1}, a_k]\big)\end{aligned}}$$

where $a_j \in \mathbb{N}$

$\rightarrow$ model for k most significant CF coefficients of $\log_b X$, analogous to model for k most significant b-ary digits of X

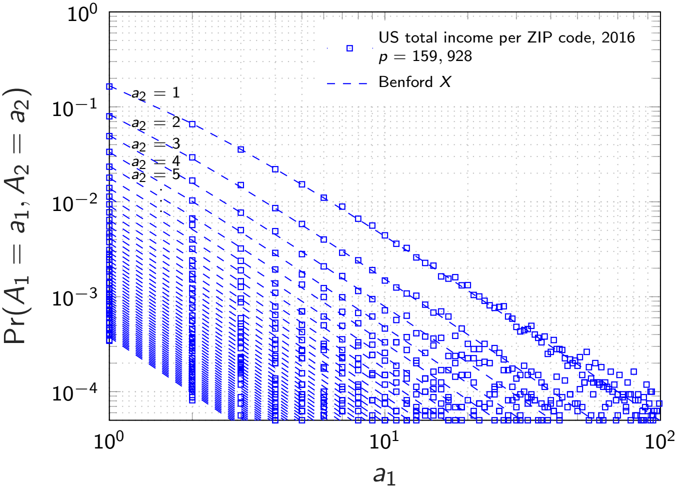
Distribution of the Two Most Significant CF Coefficients

- Pseudorandom empiricals vs theoreticals (Benford X)



CF Coefficients in Real Scale-Invariant Datasets

- Distribution of first two CF coefficients of $\log_{10} x_i$



p: dataset size

First CF Coefficient A_1 vs First Significant b-ary Digit $A_{(1)}$

A_1  $A_{(1)}$

- Which r.v. should we use in a forensic detection test where X is hypothesised to be Benford?
  a) $A_1 \;\; = \lfloor \{\log_b X\}^{-1} \rfloor$
  b) $A_{(1)} = \lfloor b^{\{\log_b X\}} \rfloor$

First CF Coefficient A_1 vs First Significant b-ary Digit $A_{(1)}$

$\boxed{A_1}$  $\cancel{A_{(1)}}$

- Which r.v. should we use in a forensic detection test where X is hypothesised to be Benford?
  - a) $A_1 \;\; = \lfloor \{\log_b X\}^{-1} \rfloor$
  - b) $A_{(1)} = \lfloor b^{\{\log_b X\}} \rfloor$
- Possible answers:
  - a) because there is <u>less information loss</u> wrt $\{Y\} = \{\log_b X\}$

$$\begin{aligned} I(A_1;\{Y\}) &= 2.046 \text{ nats} \\ I(A_{(1)};\{Y\}) &= 1.993 \text{ nats} \quad (b = 10) \end{aligned}$$

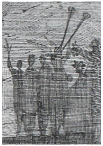
First CF Coefficient A_1 vs First Significant b-ary Digit $A_{(1)}$

$\cancel{A_1}$  $\boxed{A_{(1)}}$

- Which r.v. should we use in a forensic detection test where X is hypothesised to be Benford?
  - a) $A_1 \; = \lfloor \{\log_b X\}^{-1} \rfloor$
  - b) $A_{(1)} = \lfloor b^{\{\log_b X\}} \rfloor$
- Possible answers:
  - b) because there is less information loss wrt $\{Y\} = \{\log_b X\}$

$$
\begin{aligned}
I(A_1; \{Y\}) &= 2.046 \text{ nats} \\
I(A_{(1)}; \{Y\}) &= 2.413 \text{ nats} \quad (b = 16)
\end{aligned}
$$

First CF Coefficient A_1 vs First Significant b-ary Digit $A_{(1)}$

$\cancel{A_1}$  $\cancel{A_{(1)}}$

- Which r.v. should we use in a forensic detection test where X is hypothesised to be Benford?
  - a) $A_1 \;\; = \lfloor \{\log_b X\}^{-1} \rfloor$
  - b) $A_{(1)} = \lfloor b^{\{\log_b X\}} \rfloor$
- Possible answers:
  - $\rightarrow$ none of them: using $\{\log_b X\}$ should always be better

Time to Recap

1. The most significant digits in scale-invariant data can often be modelled using a generalisation of Benford's distribution based on heavy-tailed Pareto variables
2. There is nothing special about significant b-ary digits: they may be replaced by significant continued fraction coefficients in forensic detection tests

Time to Recap

1. The most significant digits in scale-invariant data can often be modelled using a generalisation of Benford's distribution based on heavy-tailed Pareto variables
2. There is nothing special about significant *b*-ary digits: they may be replaced by significant continued fraction coefficients in forensic detection tests
   - and both are just shadows. . .

Go raibh míle maith agaibh